SET THEORY HOMEWORK 3

Due Wednesday, October 16.

Problem 1. *Let κ be a regular uncountable cardinal.*

(1) *Suppose that $\tau < \kappa$ and $\langle C_i \mid i < \tau\rangle$ is a family of club subsets of κ. Show that $\bigcap_{i<\tau} C_i$ is also a club subset of κ.*
(2) *Let $G : \kappa \to \kappa$ be any function. Show that the set $C = \{\gamma < \kappa \mid (\forall\alpha < \gamma)(G(\alpha) < \gamma)\}$ is a club subset of κ.*

Problem 2. *Suppose that $\mathcal{F} \subset \mathcal{P}(\kappa)$ is a κ-complete normal ultrafilter on κ. I.e. $\mathcal{F}$ satisfies the following:*

- *for all $A \subset \kappa$, $A \in \mathcal{F}$ or $\kappa \setminus A \in \mathcal{F}$,*
- *if $\tau < \kappa$ and $\{A_\xi \mid \xi < \tau\}$ are sets in $\mathcal{F}$, then $\bigcap_{\xi<\tau} A_\xi \in \mathcal{F}$.*
- *for all regressive functions $f : \kappa \to \kappa$ (regressive means that $f(\alpha) < \alpha$ for all α), there is $\gamma < \kappa$ such that $f^{-1}(\gamma) := \{\alpha < \kappa \mid f(\alpha) = \gamma\}$ is in $\mathcal{F}$.*

Show that $\mathcal{F}$ is closed under diagonal intersections of length κ.

Recall that an inaccessible cardinal κ is a Mahlo cardinal if the set of regular cardinals below κ is stationary.

Problem 3. *Let κ be an inaccessible cardinal.*

(1) *Show that if κ is Mahlo, then κ is the κ-th inaccessible cardinal.*
(2) *Let κ be the least inaccessible cardinal such that κ is the κ-th inaccessible cardinal. Then κ is not Mahlo. [Hint: use $f(\lambda) = \alpha$ where λ is the α-th inaccessible.]*

Problem 4. *Assume that $V = L$. Prove that $V_\alpha = L_\alpha$ iff $\alpha = \aleph_\alpha$. Here you can use the theorem that in $V = L$, GCH holds.*

Problem 5. *Suppose that $M \prec L_{\omega_1}$. Show that M is transitive. (Hint: for $X \in M$, take the $\prec_L$-least onto $f : \omega \to X$. Show that f is definable in L_{ω_1} from X and use this to show that $f \in M$. Also show $\omega \subset M$. Use these to prove that range of f is a subset of M)*